STAT 666 FORMULA SHEET

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_p \end{bmatrix} = \begin{bmatrix} \frac{1}{n}\sum_{i=1}^n x_{i1} \\ \frac{1}{n}\sum_{i=1}^n x_{i2} \\ \vdots \\ \frac{1}{n}\sum_{i=1}^n x_{ip} \end{bmatrix} = \frac{1}{n}\mathbf{X}'\mathbf{1}$$

$$\mathbf{S} = \begin{bmatrix} s_{11} & \cdots & s_{1p} \\ s_{21} & \cdots & s_{2p} \\ \vdots & & \vdots \\ s_{p1} & \cdots & s_{pp} \end{bmatrix} = \tfrac{1}{n-1}\mathbf{X}'(\mathbf{I} - \tfrac{1}{n}\mathbf{1}\mathbf{1}')\mathbf{X},$$

$$\mathbf{D} = \text{diag}(s_{11}, s_{22}, \dots, s_{pp})$$

$$\mathbf{D}^{-1/2} = \begin{bmatrix} \left(\frac{1}{s_{11}}\right)^{1/2} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left(\frac{1}{s_{pp}}\right)^{1/2} \end{bmatrix}.$$

$$\mathbf{R} = \mathbf{D}^{-1/2}\mathbf{S}\mathbf{D}^{-1/2}$$

$$\underset{(m+q)\times(n+r)}{\mathbf{A}} = \begin{bmatrix} \underset{m\times n}{\mathbf{A}_{11}} & \underset{m\times r}{\mathbf{A}_{12}} \\ \underset{q\times n}{\mathbf{A}_{21}} & \underset{q\times r}{\mathbf{A}_{22}} \end{bmatrix} \quad \underset{(n+r)\times(s+t)}{\mathbf{B}} \begin{bmatrix} \underset{n\times s}{\mathbf{B}_{11}} & \underset{n\times t}{\mathbf{B}_{12}} \\ \underset{r\times s}{\mathbf{B}_{21}} & \underset{r\times t}{\mathbf{B}_{22}} \end{bmatrix}.$$

– Since all of the submatrices are conformable,

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} & \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} & \mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix}$$

– If $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are square ($m = n$ and $q = r$) and $\mathbf{A}_{12} = \mathbf{A}_{21}' = \mathbf{0}$,

$$|\mathbf{A}| = |\mathbf{A}_{11}|\,|\mathbf{A}_{22}|$$

– If $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are both square and nonsingular,

$$\begin{aligned} |\mathbf{A}| &= |\mathbf{A}_{11}|\left|\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right| \\ &= |\mathbf{A}_{22}|\left|\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right| \end{aligned}$$

$$\underbrace{|\mathbf{A} - \lambda\mathbf{I}_p| = 0}_{\text{“characteristic equation”}}$$

$$\mathbf{A} = \mathbf{CDC}'$$

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^p \lambda_i$$

$$|\mathbf{A}| = \prod_{i=1}^p \lambda_i$$

$$\text{var}\{\bar{\mathbf{x}}\} = \frac{1}{n}\mathbf{\Sigma}$$

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{p}{2}}|\mathbf{\Sigma}|^{\frac{1}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}$$

Constant probability density contour = $\left\{\text{all } \mathbf{x} \ni (\mathbf{x}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu) = c^2\right\}$

If $\underset{p\times 1}{\mathbf{x}} \sim N_p(\boldsymbol{\mu}, \mathbf{\Sigma})$ and $\begin{pmatrix}\mathbf{x}_1 \\ \mathbf{x}_2\end{pmatrix} \sim N_p\left(\begin{bmatrix}\boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2\end{bmatrix}, \begin{bmatrix}\mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22}\end{bmatrix}\right)$

- $\mathbf{Ax} + \mathbf{c} \sim N_q(\mathbf{A}\boldsymbol{\mu} + \mathbf{c}, \mathbf{A\Sigma A}')$
  - * $\mathbf{c}'\mathbf{x} \sim N_1(\mathbf{c}'\boldsymbol{\mu}, \mathbf{c}'\mathbf{\Sigma c})$
  - * $E\{\mathbf{\Sigma}^{-\frac{1}{2}}(\mathbf{x}-\boldsymbol{\mu})\} = \mathbf{0_p}$
    $\text{var}\{\mathbf{\Sigma}^{-\frac{1}{2}}(\mathbf{x}-\boldsymbol{\mu})\} = \mathbf{I_p}$
    and $\mathbf{\Sigma}^{-\frac{1}{2}}(\mathbf{x}-\boldsymbol{\mu}) \sim N_p(\mathbf{0}, \mathbf{I})$
- $\mathbf{x}_1|\mathbf{x}_2 \sim N(\boldsymbol{\mu}_1 + \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2), \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21})$
- $(\mathbf{x}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) \sim \chi^2_p$

Normal QQ Plot: Plot $\Phi^{-1}\left(\frac{i-\frac{1}{2}}{n}\right)$ vs. $x_{(1)}, \dots, x_{(n)}$

χ^2 QQ Plot: Plot $\left(\frac{i-\frac{1}{2}}{n}\right)^{th}$ quantile of χ^2_p vs. $D^2_{(i)} = i^{th}$ ordered value of $D_i^2 = (\mathbf{x}_i - \bar{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x}_i - \bar{\mathbf{x}})$

Box and Cox (1964) recommend using

$$x^{(\lambda)} = \begin{cases} \frac{x^\lambda - 1}{\lambda} & \text{for } \lambda \neq 0 \\ \ln(x) & \text{for } \lambda = 0 \end{cases}$$

where λ is chosen by maximizing

$$\ell(\lambda) = -\frac{n}{2}\ln\left[\frac{1}{n}\sum_{i=1}^n (x_i^{(\lambda)} - \widetilde{x^{(\lambda)}})^2\right] + (\lambda - 1)\sum_{i=1}^n \ln(x_i)$$

and $\widetilde{x^{(\lambda)}} = \frac{1}{n}\sum_{i=1}^n x_i^{(\lambda)}$

CLT: Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be independent obs. from a population with mean $\boldsymbol{\mu}$ and variance $\mathbf{\Sigma}$.

- $\sqrt{n}(\bar{\mathbf{x}} - \boldsymbol{\mu})$ is approx. $N_p(\mathbf{0}, \mathbf{\Sigma})$
- $n(\bar{\mathbf{x}} - \boldsymbol{\mu})'\mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu})$ is approx χ^2_p for $n - p$ large.

Let $\underset{p\times 1}{\mathbf{x}_i} = \begin{bmatrix}\mathbf{x}_i^{(1)} \\ \mathbf{x}_i^{(2)}\end{bmatrix}$ $\begin{matrix}\leftarrow \text{missing components } (q\times 1) \\ \leftarrow \text{observed components } ((p-q)\times 1)\end{matrix}$

$$\underset{p\times 1}{\tilde{\boldsymbol{\mu}}} = \begin{bmatrix}\tilde{\boldsymbol{\mu}}^{(1)} \\ \tilde{\boldsymbol{\mu}}^{(2)}\end{bmatrix} \text{ and } \tilde{\boldsymbol{\Sigma}} = \begin{bmatrix}\tilde{\boldsymbol{\Sigma}}_{11} & \tilde{\boldsymbol{\Sigma}}_{12} \\ \tilde{\boldsymbol{\Sigma}}_{21} & \tilde{\boldsymbol{\Sigma}}_{22}\end{bmatrix}$$

Each "E" step estimates $\mathbf{x}_i^{(1)}$ using regression:

$$\underset{q\times 1}{\tilde{\mathbf{x}}_i}^{(1)} = \underset{q\times 1}{\tilde{\boldsymbol{\mu}}_i}^{(1)} + \underset{q\times(p-q)}{\mathbf{B}}\left(\underbrace{\mathbf{x}_i^{(2)} - \tilde{\boldsymbol{\mu}}_i^{(2)}}_{(p-q)\times 1}\right)$$

where regression coefficients

$$\mathbf{B} = \underset{q\times(p-q)}{\tilde{\boldsymbol{\Sigma}}_{12}} \underset{(p-q)\times(p-q)}{\tilde{\boldsymbol{\Sigma}}_{22}^{-1}}$$

$$T^2 = n(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)'\mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0) \sim T^2_{p,\nu}$$

$$\frac{\nu - p + 1}{\nu p} T^2_{p,v} = F_{p,\nu-p+1}$$

$100(1-\alpha)\%$ C.R. for $\boldsymbol{\mu}$:

$$\{\text{all } \boldsymbol{\mu} \ni \underbrace{n(\bar{\mathbf{x}} - \boldsymbol{\mu})'\mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu})}_{\text{squared mult. distance from } \bar{\mathbf{x}}} \le T^2_{\alpha,p,\nu}\}$$

$$\mathbf{a}'\bar{\mathbf{x}} \pm t_{\frac{\alpha}{2},n-1}\sqrt{\frac{\mathbf{a}'\mathbf{S}\mathbf{a}}{n}}$$

$$\mathbf{a}'\bar{\mathbf{x}} \pm \sqrt{T^2_{\alpha,p,\nu}}\sqrt{\frac{\mathbf{a}'\mathbf{S}\mathbf{a}}{n}}$$

$$\mathbf{a}'\bar{\mathbf{x}} \pm t_{\frac{\alpha}{2k},n-1}\sqrt{\frac{\mathbf{a}'\mathbf{S}\mathbf{a}}{n}}$$

$$\mathbf{a} = \mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu})$$

$$\mathbf{a}^* = \mathbf{D}^{\frac{1}{2}}\mathbf{a} \qquad \text{where } \mathbf{D} = \begin{bmatrix} s_{11} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & s_{pp}\end{bmatrix}$$

$T^2 = n(\mathbf{C}\bar{\mathbf{x}})'(\mathbf{CSC}')^{-1}\mathbf{C}\bar{\mathbf{x}} \sim T^2_{p-1,n-1}$ (assuming $p-1$ rows in $\mathbf{C}$)

$$T^2 = [\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2]'\left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\mathbf{S}_{p\ell}\right]^{-1}[\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2] \sim T^2_{p,n_1+n_2-2}$$

$$\mathbf{a}'\bar{\mathbf{x}}_1 - \mathbf{a}'\bar{\mathbf{x}}_2 \pm t_{\frac{\alpha}{2},n_1+n_2-2}\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\mathbf{a}'\mathbf{S}_{p\ell}\mathbf{a}}$$

$$\mathbf{a}'\bar{\mathbf{x}}_1 - \mathbf{a}'\bar{\mathbf{x}}_2 \pm t_{\frac{\alpha}{2k},n_1+n_2-2}\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\mathbf{a}'\mathbf{S}_{p\ell}\mathbf{a}}$$

$$\mathbf{a}'\bar{\mathbf{x}}_1 - \mathbf{a}'\bar{\mathbf{x}}_2 \pm \sqrt{T^2_{\alpha,p,\nu}}\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\mathbf{a}'\mathbf{S}_{p\ell}\mathbf{a}}$$

$$\mathbf{a} = \mathbf{S}_{p\ell}^{-1}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)$$

$$\mathbf{a}^\star = D_{p\ell}^{\frac{1}{2}}\mathbf{a}$$

$$\mathbf{D}_{p\ell} = \text{diag}\{\mathbf{S}_{p\ell}\} = \begin{bmatrix} s_{11,p\ell} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & s_{pp,p\ell}\end{bmatrix}$$

Nel and Van der Merwe (1986) test uses

$$T^{*^2} = (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_e^{-1}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)$$

$$\mathbf{S}_e = \frac{\mathbf{S}_1}{n_1} + \frac{\mathbf{S}_2}{n_2}$$

$$T^{*^2} \overset{\text{approx}}{\sim} T^2_{p,\nu^*}$$

$$\nu^* = \frac{\text{tr}\left\{\mathbf{S}_e^2\right\} + (\text{tr}\{\mathbf{S}_e\})^2}{\sum_{i=1}^{2}\frac{1}{n_i - 1}\left[\text{tr}\left\{\left(\frac{\mathbf{S}_i}{n_i}\right)^2\right\} + \left(\text{tr}\left\{\frac{\mathbf{S}_i}{n_i}\right\}\right)^2\right]}$$

$$T^2_{\text{add}} = (\nu - p)\frac{T^2_{p+q} - T^2_p}{\nu + T^2_p} \sim T^2_{q,\nu-p}$$

$$\underbrace{\sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell}(\mathbf{x}_{\ell_j} - \bar{\mathbf{x}})(\shortparallel)'}_{\substack{\text{total corrected} \\ \text{sum of squares} \\ \text{and cross} \\ \text{products matrix}}} = \underbrace{\sum_{\ell=1}^{g} n_\ell(\bar{\mathbf{x}}_\ell - \bar{\mathbf{x}})(\shortparallel)'}_{\substack{=\mathbf{H} \\ \text{"Between"} \\ \text{groups} \\ \text{matrix}}} + \underbrace{\sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell}(\mathbf{x}_{\ell_j} - \bar{\mathbf{x}}_\ell)(\shortparallel)'}_{\substack{=\mathbf{E} \\ \text{"Within" groups} \\ \text{matrix} \\ = \sum_{\ell=1}^{g}(n_\ell - 1)\mathbf{S}_\ell}}$$

$$\Lambda = \frac{|\mathbf{E}|}{|\mathbf{E} + \mathbf{H}|} \sim \Lambda_{\alpha,p,\nu_H,\nu_E}$$

Let $\lambda_1, \ldots, \lambda_s$ be the s non-zero eigenvalues of $\mathbf{E}^{-1}\mathbf{H}$, where $s = \min(p, \nu_H)$. Then $\Lambda = \prod_{i=1}^{s}\frac{1}{1+\lambda_i}$

$$F = \frac{1 - \Lambda^{1/t}}{\Lambda^{1/t}}\;\frac{df_2}{df_1} \overset{\text{approx}}{\sim} F_{df_1,df_2}$$

$$\begin{aligned} df_1 &= p\nu_H \\ df_2 &= wt - \frac{1}{2}(p\nu_H - 2) \\ w &= \nu_E + \nu_H - \frac{1}{2}(p + \nu_H + 1) \\ t &= \begin{cases} \sqrt{\frac{p^2\nu_H^2 - 4}{p^2 + \nu_H^2 - 5}} & \text{for } p^2 + \nu_H^2 - 5 > 0 \\ 1 & \text{for } p^2 + \nu_H^2 - 5 \le 0 \end{cases} \end{aligned}$$

$$\theta = \lambda_1$$

$$V = \sum_{i=1}^{s} \frac{\lambda_i}{1+\lambda_i} = \text{tr}\left\{ (\mathbf{E}+\mathbf{H})^{-1}\mathbf{H} \right\} = \sum_{i=1}^{s} \xi_i$$

where $\xi_1, \ldots, \xi_s$ are the s ordered e'vals of $(\mathbf{E}+\mathbf{H})^{-1}\mathbf{H}$

$$\xi_i = \frac{\lambda_i}{1+\lambda_i} \text{ and } \lambda_i = \frac{\xi_i}{1-\xi_i}$$

$$U = \sum_{i=1}^{s} \lambda_i = \text{tr}\{\mathbf{E}^{-1}\mathbf{H}\}$$

$$\hat{\boldsymbol{\delta}} = c_1\bar{\mathbf{x}}_1 + c_2\bar{\mathbf{x}}_2 + \cdots + c_g\bar{\mathbf{x}}_g$$

$$\text{var}\{\hat{\boldsymbol{\delta}}\} = \sum_{i=1}^{g} {c_i}^2 \frac{\boldsymbol{\Sigma}}{n_i} = \left(\sum_{i=1}^{g} \frac{{c_i}^2}{n_i} \right) \boldsymbol{\Sigma}$$

$$\widehat{\text{var}}\{\hat{\boldsymbol{\delta}}\} = \left(\sum_{i=1}^{g} \frac{{c_i}^2}{n_i} \right) \mathbf{S}_{p\ell}$$

where $\mathbf{S}_{p\ell} = \frac{1}{\nu_E}\mathbf{E}$ and $\nu_E = \sum_{i=1}^{g}(n_i - 1)$.

$$T^2 = \hat{\boldsymbol{\delta}}' \left[\widehat{\text{var}}\left\{ \hat{\boldsymbol{\delta}} \right\} \right]^{-1} \hat{\boldsymbol{\delta}} \sim T^2_{p,\nu_E}$$

$$\mathbf{H}_1 = \frac{1}{\sum_{i=1}^{g} \frac{c_i^2}{n_i}} \hat{\boldsymbol{\delta}}\hat{\boldsymbol{\delta}}' \text{ and } \Lambda = \frac{|\mathbf{E}|}{|\mathbf{E}+\mathbf{H}_1|} \sim \Lambda_{p,1,\nu_E}$$

$100(1-\alpha)\%$ (Simultaneous) Confidence Interval for $\mu_{ij} - \mu_{kj}$ is:

$$(\bar{x}_{ij} - \bar{x}_{kj}) \pm t_{\underbrace{[\alpha/(pg(g-1))]}_{\substack{\frac{\alpha}{2} \text{ divided by} \\ \#\text{ of comparisons} \\ = pg(g-1)/2}}, [\sum_{i=1}^{g}(n_i-1)]} \sqrt{s_{p\ell,jj}\left(\frac{1}{n_i} + \frac{1}{n_k}\right)}$$

where $s_{pl,jj}$ is the j^{th} diagonal element of $\mathbf{S}_{p\ell} = \mathbf{E}/\left(\sum_{i=1}^{g}(n_i - 1)\right)$

λ_1 = largest e'value of $\mathbf{E}^{-1}\mathbf{H}$ and $\mathbf{a}_1$ is corresp. e'vec
$a_i^* = a_i\sqrt{s_{p\ell,ii}} = a_i \times \sqrt{\frac{1}{\nu_E} e_{ii}}$

$$\underset{\substack{\uparrow \\ \text{"partial } \Lambda \\ \text{statistic"}}}{\Lambda_{z|y}} = \frac{\Lambda_{yz}}{\Lambda_y} \sim \Lambda_{\underset{\substack{\uparrow \\ \#\text{ of vars in } \mathbf{z}}}{q}, \nu_H, \nu_E - \underset{\substack{\uparrow \\ \#\text{ of vars in } \mathbf{y}}}{p}}$$

$$\Lambda_A = \frac{|\mathbf{E}|}{|\mathbf{E}+\mathbf{H}_A|} \sim \Lambda_{p,a-1,ab(n-1)}$$

$$\Lambda_B = \frac{|\mathbf{E}|}{|\mathbf{E}+\mathbf{H}_B|} \sim \Lambda_{p,b-1,ab(n-1)}$$

$$\Lambda_{AB} = \frac{|\mathbf{E}|}{|\mathbf{E}+\mathbf{H}_{AB}|} \sim \Lambda_{p,(a-1)(b-1),ab(n-1)}$$

Test for H_{01}:

$$\Lambda = \frac{|\mathbf{CEC}'|}{|\mathbf{C}(\mathbf{E}+\mathbf{H})\mathbf{C}'|} \sim \Lambda_{p-1,\nu_H,\nu_E}$$

Test for H_{02}:

$$\Lambda = \frac{\mathbf{1}'\mathbf{E}\mathbf{1}}{\mathbf{1}'\mathbf{E}\mathbf{1} + \mathbf{1}'\mathbf{H}\mathbf{1}} \sim \Lambda_{1,\nu_H,\nu_E}$$

$$\Rightarrow \frac{1-\Lambda}{\Lambda}\frac{\nu_E}{\nu_H} \sim F_{\nu_H,\nu_E}$$

Test for H_{03}:

$$T^2 = \left(\sum_{\ell=1}^{g} n_\ell \right) (\mathbf{C}\bar{\mathbf{x}})' \left(\frac{1}{\nu_e}\mathbf{CEC}' \right)^{-1} \mathbf{C}\bar{\mathbf{x}} \sim T^2_{p-1,\nu_E}$$

$$\Rightarrow \frac{\nu_E - (p-1) + 1}{\nu_E(p-1)} T^2 \sim F_{p-1,\nu_E-(p-1)+1}$$

$$\varepsilon = \frac{\left[\text{tr}\left(\boldsymbol{\Sigma} - \frac{1}{p}\mathbf{1}\mathbf{1}'\boldsymbol{\Sigma} \right) \right]^2}{(p-1)\text{tr}\left[\left(\boldsymbol{\Sigma} - \frac{1}{p}\mathbf{1}\mathbf{1}'\boldsymbol{\Sigma} \right)^2 \right]}$$

SAS: "G – G ε"

Test for A: $F = \frac{MSA}{MSE} \sim F_{\hat{\varepsilon}(p-1),\hat{\varepsilon}g(n-1)(p-1)}$
Test for AB: $F = \frac{MSAB}{MSE} \sim F_{\hat{\varepsilon}(g-1)(p-1),\hat{\varepsilon}g(n-1)(p-1)}$

$$T^2 = \underset{\substack{\uparrow \\ \sum_{i=1}^{g} n_i}}{N} (\mathbf{C} \underset{\substack{\uparrow \\ \text{grand} \\ \text{mean}}}{\bar{\mathbf{x}}})'(\mathbf{C} \underbrace{\mathbf{S}_{p\ell}}_{\frac{\mathbf{E}}{\nu_E}} \mathbf{C}')^{-1}(\underbrace{\mathbf{C}}_{(p-1)\times p} \bar{\mathbf{x}}) \sim T^2_{p-1,\nu_E}$$

$$\Lambda = \frac{\mathbf{1}'\mathbf{E}\mathbf{1}}{\mathbf{1}'\mathbf{E}\mathbf{1} + \mathbf{1}'\mathbf{H}\mathbf{1}} \sim \Lambda_{1,\nu_H,\nu_E}$$

$$\Lambda = \frac{|\mathbf{CEC}'|}{|\mathbf{C}(\mathbf{E}+\mathbf{H})\mathbf{C}'|} \sim \Lambda_{p-1,\nu_H,\nu_E}$$

A and B are within subjects factors and C is a between subjects factor

Test for A:

$$T^2 = N(\mathbf{A}\,\bar{\mathbf{x}})'(\mathbf{A}\,\mathbf{S}_{p\ell}\,\mathbf{A}')^{-1}\mathbf{A}\bar{\mathbf{x}} \sim T^2_{a-1,\;\nu_E}$$

($\bar{\mathbf{x}}$: grand mean; $\mathbf{S}_{p\ell} = \frac{1}{\nu_E}\mathbf{E}$; $\nu_E = \sum_{i=1}^g (n_i - 1)$ when only one between subjects factor is used)

OR

$$\Lambda = \frac{|\mathbf{A}\mathbf{E}\mathbf{A}'|}{|\mathbf{A}(\mathbf{E}+\mathbf{H}^*)\mathbf{A}'|} \sim \Lambda_{a-1,1,\nu_E}$$

where $\mathbf{H}^* = N\bar{\mathbf{x}}\bar{\mathbf{x}}'$ and $\mathbf{A} = \mathbf{A}^* \otimes \mathbf{1}_b'$

Test for B:

$$T^2 = N(\mathbf{B}\bar{\mathbf{x}})'(\mathbf{B}\mathbf{S}_{p\ell}\mathbf{B}')^{-1}\mathbf{B}\bar{\mathbf{x}} \sim T^2_{b-1,\nu_E}$$

OR

$$\Lambda = \frac{|\mathbf{B}\mathbf{E}\mathbf{B}'|}{|\mathbf{B}(\mathbf{E}+\mathbf{H}^*)\mathbf{B}'|} \sim \Lambda_{b-1,1,\nu_E}$$

where $\mathbf{B} = \mathbf{1}_a' \otimes \mathbf{B}^*$

Test of AB interaction:

$$T^2 = N(\mathbf{G}\bar{\mathbf{x}})'(\mathbf{G}\mathbf{S}_{p\ell}\mathbf{G}')^{-1}\mathbf{G}\bar{\mathbf{x}} \sim T^2_{(a-1)(b-1),\nu_E}$$

OR

$$\Lambda = \frac{|\mathbf{G}\mathbf{E}\mathbf{G}'|}{|\mathbf{G}(\mathbf{E}+\mathbf{H}^*)\mathbf{G}'|} \sim \Lambda_{(a-1)(b-1),1,\nu_E}$$

where $\mathbf{G} = \mathbf{A}^* \otimes \mathbf{B}^*$

Test for C:

$$\Lambda = \frac{\mathbf{1}'\mathbf{E}\mathbf{1}}{\mathbf{1}'\mathbf{E}\mathbf{1} + \mathbf{1}'\mathbf{H}\mathbf{1}} \sim \Lambda_{1,\nu_H,\nu_E}$$

Tests for AC, BC, ABC interactions:

$$\Lambda = \frac{|\mathbf{A}\mathbf{E}\mathbf{A}'|}{|\mathbf{A}(\mathbf{E}+\mathbf{H})\mathbf{A}'|} \sim \Lambda_{a-1,\nu_H,\nu_E}$$

$$\Lambda = \frac{|\mathbf{B}\mathbf{E}\mathbf{B}'|}{|\mathbf{B}(\mathbf{E}+\mathbf{H})\mathbf{B}'|} \sim \Lambda_{b-1,\nu_H,\nu_E}$$

$$\Lambda = \frac{|\mathbf{G}\mathbf{E}\mathbf{G}'|}{|\mathbf{G}(\mathbf{E}+\mathbf{H})\mathbf{G}'|} \sim \Lambda_{(a-1)(b-1),\nu_H,\nu_E}$$

$$\begin{aligned}\text{vec }\mathbf{Y} &= \text{vec }(\mathbf{X}\mathbf{B}) + \text{vec }(\boldsymbol{\Xi}) \\ &= \underbrace{(\mathbf{I}_p \otimes \mathbf{X})}_{} \underbrace{\text{vec }\mathbf{B}}_{\equiv \underset{pr\times 1}{\boldsymbol{\beta}}} + \underbrace{\text{vec }(\boldsymbol{\Xi})}_{\equiv \underset{np\times 1}{\mathbf{e}}}\end{aligned}$$

($\mathbf{I}_p \otimes \mathbf{X}$: rank $= pr$ when rank($\mathbf{X}$)$=r$)

$$\underbrace{\mathbf{Y}'\mathbf{Y} - n\bar{\mathbf{y}}\bar{\mathbf{y}}'}_{\text{corrected total SS \& CP}} = \underbrace{\mathbf{Y}'\mathbf{Y} - \hat{\mathbf{B}}'\mathbf{X}'\mathbf{Y}}_{= \underset{p\ p}{\mathbf{E}}} + \underbrace{\hat{\mathbf{B}}'\mathbf{X}'\mathbf{Y} - n\bar{\mathbf{y}}\bar{\mathbf{y}}'}_{= \underset{p\times p}{\mathbf{H}}}$$

$$\Lambda = \frac{|\mathbf{E}|}{|\mathbf{E}+\mathbf{H}|} = \frac{|\mathbf{Y}'\mathbf{Y} - \hat{\mathbf{B}}'\mathbf{X}'\mathbf{Y}|}{|\mathbf{Y}'\mathbf{Y} - n\bar{\mathbf{y}}\bar{\mathbf{y}}'|} \sim \Lambda_{p,\underbrace{q}_{r-1},\underbrace{n-q-1}_{n-r}}$$

$$s = \min(p,q)$$

$$m = \frac{1}{2}(|q-p|-1)$$

$$N = \frac{1}{2}(n-q-p-2)$$

$$\frac{|\mathbf{S}|}{|\mathbf{S}_{xx}||\mathbf{S}_{yy}|}$$

$$\underset{p\times p}{\mathbf{H}_{\text{diff}}} = \hat{\mathbf{B}}'\mathbf{X}'\mathbf{Y} - \hat{\mathbf{B}}_r'\mathbf{X}_r'\mathbf{Y}$$

$$\underset{p\times p}{\mathbf{E}_{\text{full}}} = \mathbf{Y}'\mathbf{Y} - \hat{\mathbf{B}}'\mathbf{X}'\mathbf{Y}$$

$$\begin{aligned}\Lambda_{x_{q-h+1},\cdots,x_q|x_1,\cdots,x_{q-h}} &= \frac{|\mathbf{E}_{\text{full}}|}{|\mathbf{E}_{\text{full}} + \mathbf{H}_{\text{diff}}|} = \frac{|\mathbf{E}_{\text{full}}|}{|\mathbf{E}_{\text{red}}|} \\ &= \frac{|\mathbf{Y}'\mathbf{Y} - \hat{\mathbf{B}}'\mathbf{X}'\mathbf{Y}|}{|\mathbf{Y}'\mathbf{Y} - \hat{\mathbf{B}}_{\text{red}}'\mathbf{X}_{\text{red}}'\mathbf{Y}|} \\ &= \frac{\Lambda_{\text{full}}}{\Lambda_{\text{red}}} \\ &\sim \Lambda_{p,h,n-q-1}\end{aligned}$$

(h: # of xs)

$$s = \min(p,h)$$

$$m = \frac{1}{2}(|h-p|-1)$$

$$N = \frac{1}{2}(n-h-p-2)$$

$$\hat{\mathbf{B}}_i = \begin{bmatrix}\hat{\boldsymbol{\beta}}_{01} & \hat{\boldsymbol{\beta}}_{02} & \cdots & \hat{\boldsymbol{\beta}}_{0p} \\ \hat{\boldsymbol{\beta}}_{i1} & \hat{\boldsymbol{\beta}}_{i2} & \cdots & \hat{\boldsymbol{\beta}}_{ip}\end{bmatrix}$$

$$\Lambda_{x_i} = \frac{|\mathbf{Y}'\mathbf{Y} - \hat{\mathbf{B}}_i'\mathbf{X}'\mathbf{Y}|}{|\mathbf{Y}'\mathbf{Y} - n\bar{\mathbf{y}}\bar{\mathbf{y}}'|} \sim \Lambda_{p,1,n-2}$$

$$\Lambda_{x_i|x_1,\ldots,x_j} \sim \Lambda_{p,1,n-j-1}$$

$$\Lambda_{x_i|x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_m} \sim \Lambda_{p,1,n-m-1}$$

$u_1 = \mathbf{a}_1'\mathbf{y}$ and $v_1 = \mathbf{b}_1'\mathbf{x}$ are the "first canonical variates"

$$(\mathbf{S}_{yy}^{-1}\mathbf{S}_{yx}\mathbf{S}_{xx}^{-1}\mathbf{S}_{xy} - r^2\mathbf{I}_p)\mathbf{a} = \mathbf{0}$$

$$(\mathbf{S}_{xx}^{-1}\mathbf{S}_{xy}\mathbf{S}_{yy}^{-1}\mathbf{S}_{yx} - r^2\mathbf{I}_q)\mathbf{b} = \mathbf{0}$$

$$\mathbf{c}_i = \begin{bmatrix} s_{y_1} & 0 & \cdots & 0 \\ 0 & s_{y_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{y_p} \end{bmatrix} \mathbf{a}_i,$$

$$\mathbf{d}_i = \begin{bmatrix} s_{x_1} & 0 & \cdots & 0 \\ 0 & s_{x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{x_q} \end{bmatrix} \mathbf{b}_i$$

More simply, conduct analysis using $\mathbf{R}_{yy}^{-1}\mathbf{R}_{yx}\mathbf{R}_{xx}^{-1}\mathbf{R}_{xy}$ and $\mathbf{R}_{xx}^{-1}\mathbf{R}_{xy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yx}$ which have eigenvectors $\mathbf{c}_i$ and $\mathbf{d}_i$, respectively, and have eigenvalues $r_1^2, \ldots, r_s^2$.

$$\mathbf{\Lambda} = \prod_{i=1}^{s}(1 - r_i^2) \sim \Lambda_{p,q,n-1-q}$$

$$\Lambda_m = \prod_{i=m}^{s}(1 - r_i^2) \sim \Lambda_{p-m+1,q-m+1,n-m-q}$$

$$\text{or } \Lambda_{q-m+1,p-m+1,n-m-p}$$

$$[\text{since } \Lambda_{p,\nu_H,\nu_E} \stackrel{q}{=} \Lambda_{\nu_H,p,\nu_H+\nu_E-p}]$$

Test that the last k population eigenvalues (denoted $\lambda_{p-k+1}^{\text{pop}}, \ldots, \lambda_p^{\text{pop}}$) are equal

$$H_0 : \lambda_{p-k+1}^{\text{pop}} = \cdots = \lambda_p^{\text{pop}}$$

$$\bar{\lambda} = \frac{1}{k}\sum_{i=p-k+1}^{p} \lambda_i$$

$$u = \left(n - \frac{2p+11}{6}\right)\left(k \ln \bar{\lambda} - \sum_{i=p-k+1}^{p} \ln \lambda_i\right) \overset{\text{approx}}{\sim} \chi^2_{\frac{1}{2}(k-1)(k+2)}$$

$$\underset{p\times 1}{\mathbf{x}} = \underset{p\times 1}{\boldsymbol{\mu}} + \underset{p\times k}{\mathbf{\Lambda}}\,\underset{k\times 1}{\mathbf{f}} + \underset{p\times 1}{\mathbf{e}}$$

$$\begin{aligned} \text{var}\{\mathbf{x}\} &= \mathbf{\Lambda}\text{var}\{\mathbf{f}\}\mathbf{\Lambda}' + \text{var}\{\mathbf{e}\} \\ &= \mathbf{\Lambda}\mathbf{I}\mathbf{\Lambda}' + \mathbf{\Psi} \\ &= \mathbf{\Lambda}\mathbf{\Lambda}' + \mathbf{\Psi} \end{aligned}$$

$$\begin{aligned} \text{var}\{x_i\} &= [\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik}][\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik}]' + \psi_i \\ \underbrace{\sigma_{ii}}_{\substack{\uparrow \\ \text{variance for} \\ \text{variable } i}} &= \underbrace{\lambda_{i1}^2 + \lambda_{i2}^2 + \ldots + \lambda_{ik}^2}_{\substack{\uparrow \\ h_i^2 = \text{communality} \\ \text{for variable } i}} + \underbrace{\psi_i}_{\substack{\uparrow \\ \text{specific variance} \\ \text{for variable } i}} \end{aligned}$$

$$\mathbf{S} = \underset{p\times p}{\mathbf{A}}\,\underset{p\times p}{\mathbf{D}}\,\underset{p\times p}{\mathbf{A}'}$$

$$\cong \underset{p\times k}{\mathbf{A}_1}\,\underset{k\times k}{\mathbf{D}_1}\,\underset{k\times p}{\mathbf{A}_1'} \qquad \left(\begin{array}{c}\text{retaining only } k < p \text{ factors,} \\ \text{where } \mathbf{A}_1 = [\mathbf{a}_1 \ldots \mathbf{a}_k] \\ \text{and } \mathbf{D}_1 = \text{diag}(d_1 \ldots d_k)\end{array}\right)$$

$$= (\mathbf{A}_1\mathbf{D}_1^{1/2})(\mathbf{A}_1\mathbf{D}_1^{1/2})'$$

$$= \hat{\mathbf{\Lambda}}\hat{\mathbf{\Lambda}}'$$

$$\Rightarrow \hat{\mathbf{\Lambda}} = \mathbf{A}_1\mathbf{D}_1^{1/2}$$

and

$$\hat{\mathbf{\Psi}} = \mathbf{S} - \hat{\mathbf{\Lambda}}\hat{\mathbf{\Lambda}}' \qquad (\hat{\psi}_i = s_{ii} - \underbrace{\sum_{j=1}^{k} \hat{\lambda}_{ij}^2}_{\substack{h_i^2 \text{ or} \\ \text{"communality"}}})$$

$$\hat{\mathbf{\Lambda}} = \mathbf{A}_1^*\mathbf{D}_1^{*\frac{1}{2}}$$

where $\mathbf{A}^*\mathbf{D}^*\mathbf{A}^{*\prime}$ is the spectral decomposition of $\mathbf{S} - \mathbf{\Psi}^\circ$ (or $\mathbf{R} - \mathbf{\Psi}^\circ$)

$$\left(n - \frac{2p+4k+11}{6}\right)\ln\left(\frac{|\hat{\mathbf{\Lambda}}\hat{\mathbf{\Lambda}} + \hat{\mathbf{\Psi}}|}{|\mathbf{S}|}\right) \sim \chi^2_{\frac{1}{2}[(p-k)^2-(p+k)]}$$

$$(p-k)^2 \text{ must be } \geq (p+k)$$

$$k \leq \frac{1}{2}\left(2p + 1 - \sqrt{8p+1}\right)$$

$$\underset{k\times 1}{\hat{\mathbf{f}}_t} = (\underset{k\times p}{\hat{\mathbf{\Lambda}}'}\,\underset{p\times p}{\hat{\mathbf{\Psi}}^{-1}}\,\underset{p\times k}{\hat{\mathbf{\Lambda}}})^{-1}\,\underset{k\times p}{\hat{\mathbf{\Lambda}}'}\,\underset{p\times p}{\hat{\mathbf{\Psi}}^{-1}}(\underset{p\times 1}{\mathbf{x}_t} - \underset{p\times 1}{\bar{\mathbf{x}}})$$

$$\hat{\mathbf{f}}_t = \hat{\mathbf{\Lambda}}'(\hat{\mathbf{\Lambda}}\hat{\mathbf{\Lambda}}' + \hat{\mathbf{\Psi}})^{-1}(\mathbf{x}_t - \bar{\mathbf{x}})$$

$$\text{or } \hat{\mathbf{f}}_t = \hat{\mathbf{\Lambda}}'\mathbf{S}^{-1}(\mathbf{x}_t - \bar{\mathbf{x}})$$

$$\underbrace{\begin{bmatrix} \underset{(p-k)\times 1}{\mathbf{x}_1} \\ \underset{k\times 1}{\mathbf{x}_2} \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} \underset{(p-k)\times 1}{\boldsymbol{\beta}_0} \\ \underset{k\times 1}{\mathbf{0}} \end{bmatrix}}_{\boldsymbol{\mu}} + \underbrace{\begin{bmatrix} \underset{(p-k)\times k}{\mathbf{B}} \\ \mathbf{I}_k \end{bmatrix}}_{\mathbf{\Lambda}} \mathbf{f} + \mathbf{e}$$

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1k} \\ \vdots & & \\ \lambda_{(p-k),1} & \cdots & \lambda_{(p-k),k} \\ 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$t = \frac{\hat{\lambda}_{21}}{\text{s.e.}(\hat{\lambda}_{21})}$$

Saturated model:

$$df = \frac{1}{2}[(p-k)^2 - (p+k)] \qquad \leftarrow \boxed{df \text{ for a } \chi^2 \text{ GOF test}}$$

General model with maximum likelihood estimation:

$$\ell(\boldsymbol{\theta}, \mathbf{S}) = n\left(\log|\boldsymbol{\Sigma}[\boldsymbol{\theta}]| + \text{tr}\left\{\mathbf{S}\left(\boldsymbol{\Sigma}[\boldsymbol{\theta}]\right)^{-1}\right\} - \log|\mathbf{S}| - p\right)$$

$$\ell(\hat{\boldsymbol{\theta}}_{ML}; \mathbf{S}) \to \chi^2_{\frac{1}{2}p(p+1)-q}$$

is a goodness-of-fit statistic where q is # of params in model and $\underbrace{\frac{1}{2}p(p+1)}_{=p^*}$ is # of statistics in $\mathbf{S}$.

$$CFI = 1 - \frac{\max(\chi^2_M - df_M, 0)}{\max(\chi^2 - df_B, \chi^2_M - df_M, 0)}$$

$$RMSEA = \sqrt{\frac{1}{n-1}\frac{\max(\chi^2_M - df_M, 0)}{df_M}}$$

$$SRMR = \sqrt{\frac{2}{p(p+1)}\sum_{i=1}^{p}\sum_{j=1}^{i}\frac{(s_{ij} - \sigma_{ij})^2}{s_{ii}s_{jj}}}$$

$$\mathbf{a} = \underset{\mathbf{b}\neq\mathbf{0}}{\text{argmax}}\frac{[\mathbf{b}'(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)]^2}{\mathbf{b}'\mathbf{S}_{p\ell}\mathbf{b}}$$

$$\mathbf{a}_i^* = \mathbf{D}^{1/2}\mathbf{a}_i,\ i - 1, \ldots, s$$

$$\text{where } \mathbf{D} = \frac{1}{\nu_E}\begin{bmatrix} e_{11} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & e_{pp}\end{bmatrix} = \frac{1}{\nu_E}\text{diag}(\mathbf{E})$$

$$\text{ECM} = c\{2|1\} \cdot \Pr\{2|1\} \cdot p_1 + c\{1|2\} \cdot \Pr\{1|2\} \cdot p_2$$

ECM is minimized with the rule:

$$\hat{G}_1 = \{\mathbf{x} : p_1 f_1(\mathbf{x})\ c\{2|1\} > p_2 f_2(\mathbf{x}) c\{1|2\}\}$$

and

$$\hat{G}_2 = \{\mathbf{x} : p_1 f_1(\mathbf{x})\ c\{2|1\} < p_2 f_2(\mathbf{x}) c\{1|2\}\}$$

Assign $\mathbf{x}_0$ to $\hat{G}_1$ if:
$z_0 = \mathbf{a}'\mathbf{x}_0$ is closer to $\bar{z}_1 = \mathbf{a}'\bar{\mathbf{x}}_1$ than $\bar{z}_2 = \mathbf{a}'\bar{\mathbf{x}}_2$
or
$z_0 > \frac{1}{2}(\bar{z}_1 + \bar{z}_2)$

So that the rule is:

$$\hat{G}_1 = \left\{\mathbf{x} : (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{p\ell}^{-1}\mathbf{x} > \frac{1}{2}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{p\ell}^{-1}(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)\right\}$$

and

$$\hat{G}_2 = \left\{\mathbf{x} : (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{p\ell}^{-1}\mathbf{x} < \frac{1}{2}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{p\ell}^{-1}(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)\right\}$$

Incorporating costs and priors, rule is:

$$\hat{G}_1 = \left\{\mathbf{x} : (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{p\ell}^{-1}\mathbf{x} > \frac{1}{2}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{p\ell}^{-1}(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2) + \ln\frac{p_2\ c\{1|2\}}{p_1\ c\{2|1\}}\right\}$$

and

$$\hat{G}_2 = \left\{\mathbf{x} : (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{p\ell}^{-1}\mathbf{x} < \frac{1}{2}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{p\ell}^{-1}(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2) + \ln\frac{p_2\ c\{1|2\}}{p_1\ c\{2|1\}}\right\}$$

Assign $\mathbf{x}$ to $\hat{G}_{i'}$ if:

$$L_i(\mathbf{x}) = \bar{\mathbf{x}}_i'\mathbf{S}_{p\ell}^{-1}\mathbf{x} - \frac{1}{2}\bar{\mathbf{x}}_i'\mathbf{S}_{p\ell}^{-1}\bar{\mathbf{x}}_i, \quad i = 1, \ldots, g$$

is maximized when $i = i'$

For known priors, use:

$$L_i^*(\mathbf{x}) = \bar{\mathbf{x}}_i'\mathbf{S}_{p\ell}^{-1}\mathbf{x} - \frac{1}{2}\bar{\mathbf{x}}_i'\mathbf{S}_{p\ell}^{-1}\bar{\mathbf{x}}_i + \ln p_i, \quad i = 1, \ldots, g$$

Quadratic functions:

$$D_i^2(\mathbf{x}) = (\mathbf{x} - \bar{\mathbf{x}}_i)'\mathbf{S}_i^{-1}(\mathbf{x} - \bar{\mathbf{x}}_i)$$

or

$$Q_i(\mathbf{x}) = \ln p_i - \frac{1}{2}\ln|\mathbf{S}_i| - \frac{1}{2}(\mathbf{x} - \bar{\mathbf{x}}_i)'\mathbf{S}_i^{-1}(\mathbf{x} - \bar{\mathbf{x}}_i)$$

Assign $\mathbf{x}$ to $\hat{G}_i$ if $k_i = \underset{j}{\max} k_j$
Rule assumes $p_i = \frac{n_i}{n}$, $i = 1, \ldots, g$. If $p_i \neq \frac{n_i}{n}$, consider the revised rule:

Assign $\mathbf{x}$ to $\hat{G}_i$ if $\frac{k_i p_i}{n_i} = \underset{j}{\max}\frac{k_j p_j}{n_j}$

$$I_A = \sum_{i=1}^{k} p_{i|A}(1 - p_{i|A})$$

$$p_{i|A} = \frac{p_i(n_{iA}/n_i)}{\sum_{i=1}^{k} p_i(n_{iA}/n_i)}$$

$$p_A = \sum_{i=1}^{k} p_i(n_{iA}/n_i)$$

$$\Delta I = p_A I_A - (p_{A_L} I_{A_L} + p_{A_R} I_{A_R})$$

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(\mathbf{x} - \mathbf{y})'(\mathbf{x} - \mathbf{y})}$$

✓(a) $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD})$

✓(b) $\text{vec}\ (\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A})\text{vec}\ \mathbf{B}$

(c) $\text{tr}\{\mathbf{AB}\} = (\text{vec}\ \mathbf{A}')'\ \text{vec}\ \mathbf{B} = (\text{vec}\ \mathbf{A})'\ \text{vec}\ \mathbf{B}'$

(d) $\text{tr}\{\mathbf{ABCD}\} = (\text{vec}\ \mathbf{A}')'\ (\mathbf{D}' \otimes \mathbf{B})\ \text{vec}\ \mathbf{C} = (\text{vec}\ \mathbf{A})'\ (\mathbf{B} \otimes \mathbf{D}')\ \text{vec}\ \mathbf{C}'$

✓(e) $(\mathbf{A} \otimes \mathbf{B})' = \mathbf{A}' \otimes \mathbf{B}'$

✓(f) $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$